ON THE NONLINEAR STEADY STATE TRANSVERSE VIBRATION OF A CANTILEVER COLUMN WITH A SINUSOIDAL LONGITUDINAL END DISPLACEMENT

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James Albert Shiflett, Jr.

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LIST OF PRINCIPAL SYMBOLS

A	Steady state amplitude divided by spacial mode function
A_b	Amplitude of base displacement
A_c	Cross sectional area of column
A_ϵ	Potential energy due to column weight
B, C	Amplitude constants in assumed periodic solution, $A^2 = B^2 + C^2$
E	Young's modulus of column
ϵ	Normal strain in the x-direction
g	Acceleration due to gravity
I	Area moment of inertia of the column about a centroidal line parallel to the y axis
ℓ	Length of column
$M(x,t)$	Bending moment in the y direction
n	Integer referring to the nth mode of free vibration
$N(x,t)$	Resultant normal force at each cross section
P	Coordinate of strain gage in the x direction
$Q(x,t)$	Resultant shear force at each cross section
$s(x,t)$	Distance along deformed or undeformed neutral axis
t	Time
$T(t)$	Time dependent amplitude of column divided by spacial mode function
T	Kinetic energy of the column
$u(x,t)$, $v(x,t)$, $w(x,t)$	Displacements of a point on the column in the x, y, and z directions respectively

LIST OF PRINCIPAL SYMBOLS (Continued)

V	Strain energy of the column
x, y, z	Coordinates defining the position of particles relative to one end of the undeformed column
$\alpha, \beta, \gamma, \delta, \zeta, \varkappa, \xi$	Coefficients in nonlinear ordinary differential equation determined by Galerkin analysis
$\delta(\ )$	First variation of ( )
ϵ	Strain of middle surface
$\theta(x,t)$	Slope of column
$\xi(x,t)$	$x + u(x,t)$
ρ	Mass per unit volume of column
σ	Normal stress on planes originally parallel to the yz plane
$\psi_n(x)$	Transverse vibration
ω_n	nth natural frequency of free transverse vibration
Ω	Frequency of base displacement

Dots over variables indicate differentiation with respect to time.

SUMMARY

The nonlinear transverse response of a cantilever column excited at one end by a sinusoidal longitudinal displacement was investigated analytically and experimentally. Large deflection theory was used to write energy expressions for the application of Hamilton's principle. Two coupled nonlinear partial differential equations resulted. Simplifications were made to reduce the coupled equations to a single nonlinear partial differential equation in terms of the transverse displacement. A one term Galerkin analysis using the normal modes of free vibration of a cantilever beam was used to approximate the solution of the equation of motion. A nonlinear ordinary differential equation in time resulted. The assumption of periodic motion and the application of the Ritz Averaging method yielded two coupled nonlinear algebraic equations which could be separated by direct substitution. The substitution yielded an eighth order algebraic equation for the amplitude which was factored to the product of two fourth order equations. Each was easily solved by the quadratic formula.

Maximum strains for a single point on the column were measured experimentally and the analytic solution was used to calculate strains to compare results. Comparison showed good agreement between analytical and experimental results.

CHAPTER I

INTRODUCTION

A Statement of the Problem

This investigation examines the nonlinear transverse response of a cantilever column excited at one end by a sinusoidal longitudinal displacement. Figure 1 shows the column with rectangular cross section and the excitation.

Experiments indicate that steady state amplitude is related to the forcing frequency by a softening curve such as ND shown in Figure 2. In nonlinear forced vibration problems the assumption of a harmonic solution to approximate the real response curve ND yields two equations for the relation of amplitude and frequency. These equations plot into curves such as KC and ND in Figure 2. For values of the forcing frequency less than C in Figure 2 there are three possible solutions for the steady state amplitude. These are the identically zero solution, the solution with amplitude on curve KC, and the solution with amplitude on curve ND. The solution with steady state amplitude on curve KC is never found in experiments and can be shown to be an unstable analytical solution to the problem. Cunningham [1], for example, shows a solution of this type to be unstable.

Burnside [2] was able to determine the region of frequencies CD in Figure 2 where steady state vibrations arise due finite perturbations in the transverse direction no matter how small they may be. However,

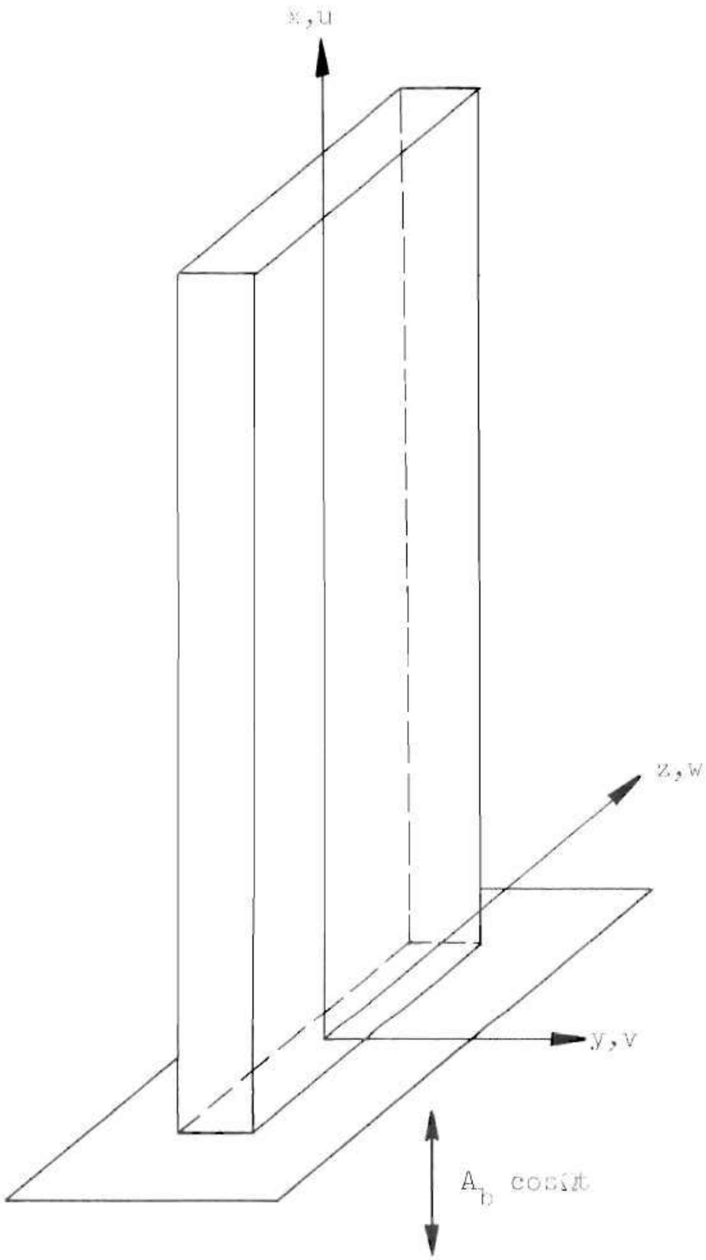

Figure 1. Column System with End Excitation.

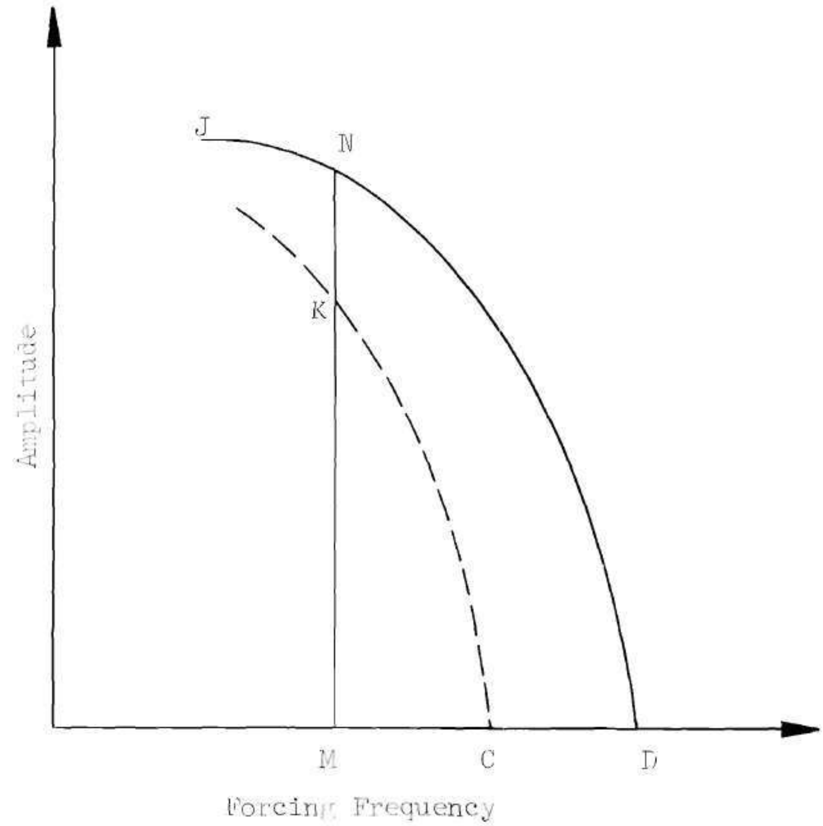

Figure 2. Amplitude Frequency Curves.

no attempt was made to determine the amplitude curves KC and ND. The purpose of this investigation is to determine the curves KC and ND analytically and to determine ND experimentally. With these curves known it can be seen that steady state vibrations can exist for values of the forcing frequency which are outside of the region CD. For forcing frequency values less than C, say M for example, steady state vibrations can exist if the perturbation given to the column is large enough. There will be a shifting of the response amplitude from the identically zero stable solution with frequency M to the finite amplitude stable solution with magnitude of the steady state amplitude equal to MN. For forcing frequencies that are less than M a larger perturbation is necessary to cause a shifting of the amplitude response from the identically zero solution to a finite amplitude response.

The shifting of stable response amplitudes described above can also take place in the reverse direction. If the column is vibrating with steady state amplitude MN and a sufficiently large transverse perturbation is given the response will shift to the identically zero solution.

A Brief Review of the Literature

Most of the work done on the stability of elastic columns has been concerned with determining frequency regions in which a given form of motion becomes dynamically unstable. In Burnside's work [2] this meant the straight configuration of the column became unstable. These investigations use small deflection theory to obtain a Mathieu or Hill equation from which the stability can be predicted but no solution is found for the amplitudes of vibration. For a brief review of the literature see Burnside [2].

According to Bolotin [3], the idea of the inadequacy of the linear treatment for determining amplitudes in the unstable region was first clearly formulated by Gol'denblat [4]. The presentation of nonlinear theory applicable to the problem of the dynamic stability of a compressed rod was given by Bolotin [3] and an analogous problem was examined almost simultaneously by Weidenhammer [5]. The book by Bolotin [3] contains much analytical work on the nonlinear problem as well as some experimental verification of the accuracy of analytical solutions. The nonlinear effects of longitudinal inertia, elasticity, and damping were considered. A paper of interest by Evensen and Evan-Iwanowski [6] was concerned with the nonlinear effect of longitudinal inertia on the response of a pinned-pinned compressed column.

Method of Solution

The derivation of the governing differential equations for the nonlinear vibration of the column was based on a paper by Eringen [7]. Figure 3 on page 8 shows an element of the column. Using the parameters shown the governing equations were obtained through Hamilton's Principle. Two coupled nonlinear partial differential equations resulted involving u, v, x, and t. In Chapter II some simplifications were made which uncoupled the equations and the resulting nonlinear partial differential equation involved only v(x,t) and derivatives. The application of a one term Galerkin type solution separated the variables and yielded a nonlinear ordinary differential equation with time as the independent variable. The assumption of periodic motion and the application of the Ritz averaging method to approximate the solution of the nonlinear ordinary differential equation yielded an algebraic equation in terms of the steady state amplitude.

An experiment was performed to determine the accuracy of the assumptions made in Chapter II and the linearizations imposed in the analysis.

CHAPTER II

ANALYTICAL SOLUTION FOR AMPLITUDE OF STEADY STATE NONLINEAR VIBRATION

The following assumptions were made:

(1) Strain energy due to shear is small compared to strain energy due to stresses on planes originally parallel to the yz plane.

(2) Kinetic energy due to rotation of column elements is small compared to the kinetic energy of translation.

(3) Strain of the middle surface is neglected compared to one.

(4) The column material obeys Hooke's law.

(5) Plane cross sections remain plane.

Assumptions (1) and (2) are due to the type of beam considered in this investigation. Timoshenko [8] has shown that these effects are small for beams vibrating at their lower natural frequencies when depth to length ratios are small.

An element of a thin column is shown in Figure 3. In terms of the coordinates shown the radius of curvature can be approximated as

$$\frac{1}{R} = \frac{\dfrac{\partial^2 v}{\partial x^2}}{\sqrt{1 - \left(\dfrac{\partial v}{\partial x}\right)^2}}$$

which is exact for inextensible motion. The strain is

$$e = \epsilon - y\left(\frac{1}{R}\right)$$

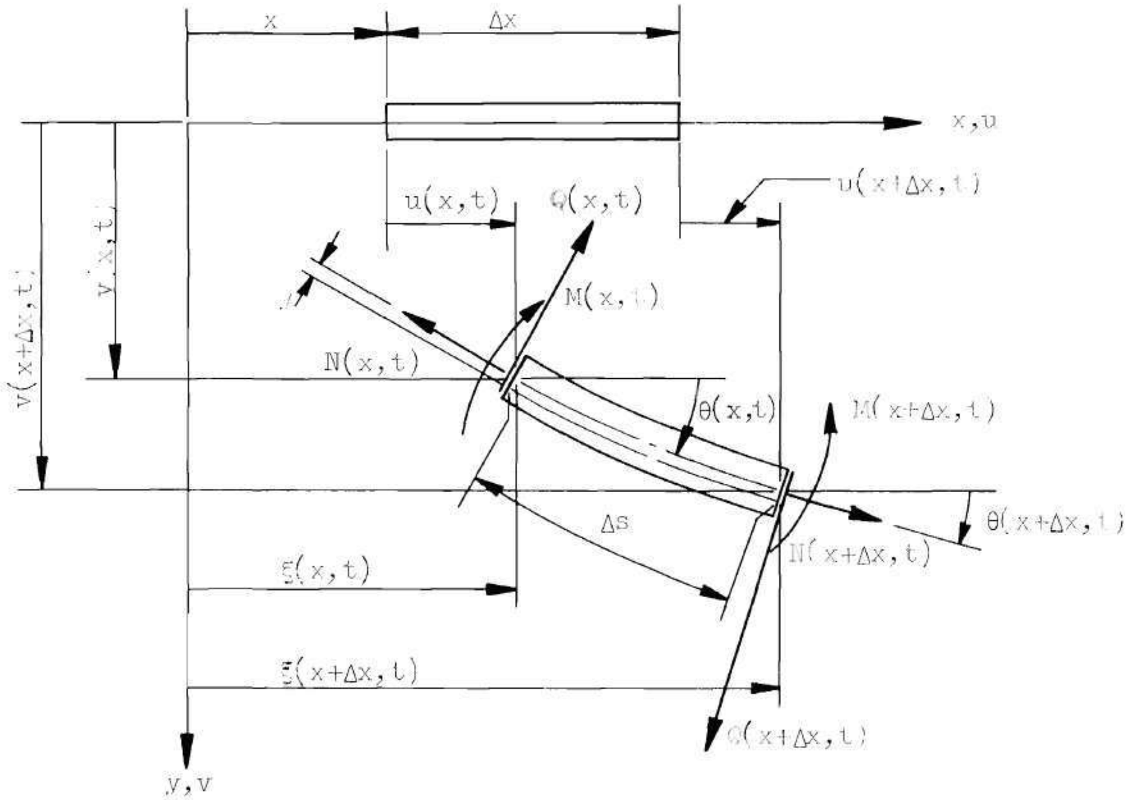

Figure 3. Element of Column.

where ϵ is the strain of the middle surface and is approximated as

$$\epsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 .$$

The strain energy is given by

$$V = \frac{EA_c}{2}\int_0^{\ell} \epsilon^2 dx + \frac{EI}{2}\int_0^{\ell}\left(\frac{1}{R}\right)^2 dx .$$

The kinetic energy is

$$T = \frac{\rho A_c}{2}\int_0^{\ell}(\dot{\xi}^2 + \dot{v}^2)dx .$$

And finally the potential energy of the external load due to the column weight is

$$A_e = \int_0^{\ell} \rho A_c g\ \xi(x,t)dx .$$

Hamilton's Principle requires

$$\delta\int_{t_o}^{t_1}(V - T + A_e)dt = 0 .$$

i.e.

$$\delta\int_{t_o}^{t_1}\int_0^{\ell}\left[\frac{EI}{2}\left(\frac{\frac{\partial^2 v}{\partial x^2}}{\sqrt{1-\left(\frac{\partial v}{\partial x}\right)^2}}\right)^2 + \frac{EA_c}{2}\epsilon^2 - \frac{\rho A_c}{2}(\dot{u}^2+\dot{v}^2) + \rho A g[x+u(x,t)]\right] dxdt = 0 . \tag{1}$$

Performing the variations in equation (1) yields the following coupled nonlinear partial differential equations:

$$\frac{EI\,\frac{\partial^4 v}{\partial x^4}}{\left[1-\left(\frac{\partial v}{\partial x}\right)^2\right]} + \frac{EI\left(\frac{\partial^2 v}{\partial x^2}\right)^3}{\left[1-\left(\frac{\partial v}{\partial x}\right)^2\right]^2} + \frac{4EI\,\frac{\partial v}{\partial x}\,\frac{\partial^2 v}{\partial x^2}\,\frac{\partial^3 v}{\partial x^3}}{\left[1-\left(\frac{\partial v}{\partial x}\right)^2\right]^2} + \frac{4EI\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial^2 v}{\partial x^2}\right)^3}{\left[1-\left(\frac{\partial v}{\partial x}\right)^2\right]^3}$$

$$- \frac{\partial}{\partial x}\left[EA_c\,\frac{\partial v}{\partial x}\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right\}\right] + \rho A_c\,\frac{\partial^2 v}{\partial t^2} = 0 \qquad (2)$$

$$- \frac{\partial}{\partial x}\left[EA_c\left\{\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right\}\right] + \rho A g + \rho A\,\frac{\partial^2 u}{\partial t^2} = 0\ . \qquad (3)$$

For the problem considered here the above equations, (2) and (3), will be uncoupled as follows:

$$N(x,t) = \int \sigma\, dA$$

$$= \int E\left(\varepsilon - \frac{y}{R}\right) dA$$

$$= EA_c\,\varepsilon = EA_c\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right].$$

From the coupling term in equation (2),

$$- \frac{\partial}{\partial x}\left[EA_c\,\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right)\right] = - N\,\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}\left[EA_c\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)\right]\frac{\partial u}{\partial x}\ . \qquad (4)$$

From (3),

$$- \frac{\partial}{\partial x}\left[EA_c\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right)\right] = - \rho A_c g - \rho A_c\,\frac{\partial^2 u}{\partial t^2}\ . \qquad (5)$$

Hence substituting (5) into (4) yields

$$-\frac{\partial}{\partial x}\left[EA_c\,\frac{\partial v}{\partial x}\left(\frac{\partial y}{\partial x}+\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right)\right]=-N\,\frac{\partial^2 v}{\partial x^2}-mg\,\frac{\partial v}{\partial x}-m\,\frac{\partial^2 u}{\partial t^2}\,\frac{\partial v}{\partial x} \qquad (6)$$

where $m = \rho A_c$ = mass per unit length. Equation (5) can be written as

$$\frac{\partial N}{\partial x} = mg + m\,\frac{\partial^2 u}{\partial t^2}\ . \qquad (7)$$

Integrating (7) and noting that for $x = 0$,

$$N(0,t) = -\left[mg\ell + \int_0^{\ell} m\,\frac{\partial^2 u}{\partial t^2}\,dx\right],$$

the relation for N is found, i.e.

$$-N(x,t) = mg(\ell - x) + m\int_x^{\ell}\frac{\partial^2 u}{\partial t^2}\,dx\ . \qquad (8)$$

Substituting (8) into (6) and in turn substituting (6) into (2) yields

$$\frac{EI\,\dfrac{\partial^4 v}{\partial x^4}}{\left[1-\left(\dfrac{\partial v}{\partial x}\right)^2\right]}+\frac{EI\left(\dfrac{\partial^2 v}{\partial x^2}\right)^3}{\left[1-\left(\dfrac{\partial v}{\partial x}\right)^2\right]^2}+\frac{4EI\,\dfrac{\partial v}{\partial x}\,\dfrac{\partial^2 v}{\partial x^2}\,\dfrac{\partial^3 v}{\partial x^3}}{\left[1-\left(\dfrac{\partial v}{\partial x}\right)^2\right]^2}+\frac{4EI\left(\dfrac{\partial v}{\partial x}\right)^2\left(\dfrac{\partial^2 v}{\partial x^2}\right)^3}{\left[1-\left(\dfrac{\partial v}{\partial x}\right)^2\right]^3}$$

$$-mg\,\frac{\partial v}{\partial x}-m\,\frac{\partial^2 u}{\partial t^2}\,\frac{\partial v}{\partial x}+mg(\ell-x)\frac{\partial^2 v}{\partial x^2}$$

$$+m\left(\int_x^{\ell}\frac{\partial^2 u}{\partial t^2}\,dx\right)\frac{\partial^2 v}{\partial x^2}+m\,\frac{\partial^2 v}{\partial t^2}=0\ . \qquad (9)$$

If nonlinear bending terms are neglected in equation (9) the following equation results:

$$EI \frac{\partial^4 v}{\partial x^4} - mg \frac{\partial v}{\partial x} - m \frac{\partial^2 u}{\partial t^2} \frac{\partial v}{\partial x} + mg(\ell - x) \frac{\partial^2 v}{\partial x^2}$$

$$+ m \left[\int_x^\ell \frac{\partial^2 u}{\partial t^2} dx \right] \frac{\partial^2 v}{\partial x^2} + m \frac{\partial^2 v}{\partial t^2} = 0 \ . \tag{10}$$

Except for a missing end mass term, equation (10) was investigated for a pinned-pinned column by Evan-Iwanowski and Evensen [6]. Equation (9) which is investigated here contains nonlinear bending terms which should yield a better approximation for the amplitudes of vibration than equation (10). The magnitude of the slope of the vibrating column is not neglected compared to one in equation (9) as it is in equation (10).

To eliminate $u(x,t)$ from equation (9) the inextensibility condition is imposed. Based on Figure 3 and the paper by Eringen [7] it can be shown that

$$ds^2 = d\xi^2 + dv^2 \qquad \text{and} \qquad ds^2 = (1+2\epsilon)dx^2 \ .$$

Neglecting ϵ yields

$$d\xi^2 = ds^2 - dv^2 \qquad \text{and} \qquad d\xi = \sqrt{1 - \left(\frac{\partial v}{\partial x}\right)^2} \ dx$$

and upon integration,

$$\xi(x,t) = \int_o^x \sqrt{1 - \left(\frac{\partial v}{\partial x}\right)^2} \ dx + C \ .$$

Expanding the radical and neglecting higher order terms gives

$$\xi(x,t) = \int_0^x \left[1 - \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] dx + A_b \cos\Omega t$$

where the term $A_b \cos\Omega t$ is the longitudinal displacement of the clamped end. Then noting that

$$\xi - x = u \ ,$$

the following expression for u results:

$$u(x,t) = A_b \cos\Omega t - \frac{1}{2} \int_0^x \left(\frac{\partial v}{\partial x}\right)^2 dx \tag{11}$$

and differentiating with respect to time yields

$$\frac{\partial^2 u}{\partial t^2} = - A_b \Omega^2 \cos\Omega t - \frac{1}{2} \frac{\partial^2}{\partial t^2} \int_0^x \left(\frac{\partial v}{\partial x}\right)^2 dx \ . \tag{12}$$

After substituting equation (12) into equation (9) and expanding the denominators of equation (9), with terms of order greater than five being neglected, the equation of motion in uncoupled form is:

$$\begin{aligned} &EI \frac{\partial^4 v}{\partial x^4} + EI \left(\frac{\partial v}{\partial x}\right)^4 \frac{\partial^4 v}{\partial x^4} + EI \left(\frac{\partial^2 v}{\partial x^2}\right)^3 + 4EI \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} \frac{\partial^3 v}{\partial x^3} \\ &\quad + 8EI \left(\frac{\partial v}{\partial x}\right)^3 \frac{\partial^2 v}{\partial x^2} \frac{\partial^3 v}{\partial x^3} + 6EI \left(\frac{\partial v}{\partial x}\right)^2 \left(\frac{\partial^2 v}{\partial x^2}\right)^3 - mg \frac{\partial v}{\partial x} \\ &\quad + mg(\ell-x) \frac{\partial^2 v}{\partial x^2} + [mA_b \Omega^2 \cos\Omega t] \frac{\partial v}{\partial x} + \frac{m}{2}\left[\frac{\partial^2}{\partial t^2} \int_0^x \left(\frac{\partial v}{\partial x}\right)^2 dx\right] \frac{\partial v}{\partial x} \\ &\quad - [m(\ell-x)A_b \Omega^2 \cos\Omega t] \frac{\partial^2 v}{\partial x^2} - \frac{m}{2}\left[\int_x^\ell \frac{\partial^2}{\partial t^2}\left[\int_0^x \left(\frac{\partial v}{\partial x}\right)^2 dx\right] dx\right] \frac{\partial^2 v}{\partial x^2} + m \frac{\partial^2 v}{\partial t^2} \\ &\quad = 0 \ . \end{aligned} \tag{13}$$

Based on experimental observation, a Galerkin type solution of the form

$$v(x,t) = \psi_n(x)T(t)$$

is assumed with n denoting vibration in a particular spacial mode which is approximated here by the normal mode of free vibration of a cantilevered beam with no end excitation. In the assumed solution

$$\psi_n(x) = \cosh \beta_n x - \cos \beta_n x - \alpha_n(\sinh \beta_n x - \sin \beta_n x)$$

are the normal modes of free vibration of a cantilever beam as tabulated by Young and Felgar [9].

The calculation of Galerkin's method is

$$\int_0^\ell \overline{DE}\ \psi_n(x)dx = 0 \qquad (14)$$

where $\overline{DE}$ is the residual obtained by substituting the assumed solution into equation (13). The calculation of (14) yields the nonlinear ordinary differential equation

$$\alpha\ddot{T} + [\beta + \gamma A_b \Omega^2 \cos\Omega t]T + \delta T^3 + \xi T^5 + \kappa T[T\ddot{T} + (\dot{T})^2] = 0\ , \qquad (15)$$

where

$$\alpha = m \int_0^\ell \psi_n^2(x)dx$$

$$\beta = \int_0^\ell [EI\ \psi_n \psi_n^{iv} - mg\ \psi_n \psi_n' + mg(\ell - x)\psi_n \psi_n'']dx$$

$$\gamma = \int_0^\ell [m\psi_n\psi_n' - m(\ell-x)\psi_n\psi_n'']dx$$

$$\delta = EI\int_0^\ell [(\psi_n')^2 \psi_n^{iv}\psi_n + \psi_n(\psi_n'')^3 + 4\psi_n\psi_n'\psi_n''\psi_n''']dx$$

$$\xi = EI\int_0^\ell [6\psi_n(\psi_n')^2(\psi_n'')^3 + 8\psi_n(\psi_n')^3\psi_n''\psi_n''' + \psi_n(\psi_n')^4\psi_n^{iv}]dx$$

$$\varkappa = m\int_0^\ell \left[\int_0^x (\psi_n')^2 dx\right]\psi_n\psi_n'\, dx - m\int_0^\ell\left[\int_x^\ell \int_0^x (\psi_n')^2\, dxdx\right]\psi_n\psi_n''\, dx .$$

The constants α, β, and γ can be evaluated by use of the integral tables of R. P. Felgar [10]. To determine the constants δ, ξ, and $\varkappa$ a combination of direct and numerical integration is required.

Equation (15) minus the term ξT^5 has been investigated by Bolotin [5] for a pinned-pinned column with damping terms included. Hence a linear damping term is added at this point as a crude approximation to the actual damping of the system. The damping term is

$$2\zeta\dot{T} ,$$

where ρ is to be determined experimentally.

Equation (15) then has the form

$$\alpha\ddot{T} + 2\zeta\dot{T} + [\beta + \gamma A_b\Omega^2\cos\Omega t]T + \delta T^3 + \xi T^5 + \varkappa T[T\ddot{T} + (\dot{T})^2] = 0. \quad (16)$$

If the change of variable

$$r = \frac{\Omega}{2}\, t$$

is employed and the following constants defined:

$$f = \frac{4\zeta}{\alpha\Omega}, \qquad a = \frac{4\beta}{\alpha\Omega^2},$$

$$b = \frac{4\gamma}{\alpha}, \qquad c = \frac{4\delta}{\alpha\Omega^2},$$

$$d = \frac{4\xi}{\alpha\Omega^2}, \qquad h = \frac{\kappa}{\alpha},$$

then

$$\ddot{T} + f\dot{T} + (a + bA_b\cos\Omega t)T + cT^3 + dT^5 + hT[T\ddot{T} + (\dot{T})^2] = 0 . \quad (17)$$

To approximate the solution of equation (17) assume

$$T(\tau) = B \sin \tau + C \cos \tau$$

and use the Ritz Averaging method. See [11]. That is

$$\int_0^{2\pi} \overline{ODE} \ \sin\tau \, d\tau = 0$$

$$\int_0^{2\pi} \overline{ODE} \ \cos\tau \, d\tau = 0$$

where $\overline{ODE}$ is the residual obtained by substituting the assumed solution into equation (17). Carrying out the integrations yields the following set of coupled nonlinear algebraic equations:

$$\left[A^4 + \frac{2}{5d}(3c-2h)A^2 + \frac{8}{5d}\left(a-1-\frac{bA_b}{2}\right)\right]B - \frac{8f}{5d}C = 0 \quad (18)$$

$$\frac{8f}{5d}B + \left[A^4 + \frac{2}{5d}(3c-2h)A^2 + \frac{8}{5d}\left(a-1+\frac{bA_b}{2}\right)\right]C = 0 \quad (19)$$

where $A^2 = B^2 + C^2$. The assumed solution

$$T(\tau) = B \sin\tau + C \cos\tau$$

can be written in the form

$$T(\tau) = A \sin(\tau+\varphi)$$

where $A = \sqrt{B^2+C^2}$ and $\tan\varphi = \frac{C}{B}$. If equations (18) and (19) are written in terms of A and φ two equations for the angle φ result:

$$\tan\varphi = \frac{5d}{8f}\left[A^4 + \frac{2}{5d}(3c-2h)A^2 + \frac{8}{5d}\left(a-1-\frac{bA_b}{2}\right)\right] \tag{18a}$$

and

$$\tan\varphi = -\frac{8f}{5d}\left[A^4 + \frac{2}{5d}(3c-2h)A^2 + \frac{8}{5d}\left(a-1+\frac{bA_b}{2}\right)\right] . \tag{19a}$$

Equating the two expressions for $\tan\varphi$ and substituting for the constants a, b, c, d, h, and f gives

$$\begin{aligned}&25\xi^2A^8 + 10\xi(6\delta-\varkappa\Omega^2)A^6 + [20\xi(4\beta-\alpha\Omega^2) + (6\delta-\varkappa\Omega^2)^2]A^4\\&\quad + [4(4\beta-\alpha\Omega^2)(6\delta-\varkappa\Omega^2)]A^2\\&\quad + 4[4\beta-(\alpha+2\gamma A_b)\Omega^2][4\beta-(\alpha-2\gamma A_b)\Omega^2] + 64\zeta^2\Omega^2 = 0 .\end{aligned} \tag{20}$$

Equation (20) factors to

$$\begin{aligned}&[5\xi A^4 + (6\delta-\varkappa\Omega^2)A^2 + [2(4\beta-\alpha\Omega^2)\\&\quad + 4\Omega\sqrt{\gamma^2A_b^2\Omega^2-4\zeta^2}]][5\xi A^4 + (6\delta-\varkappa\Omega^2)A^2\\&\quad + [2(4\beta-\alpha\Omega^2) - 4\Omega\sqrt{\gamma^2A_b^2\Omega^2-4\zeta^2}]] = 0 .\end{aligned} \tag{21}$$

Equation (21) then yields two amplitude frequency relations:

$$A_1 = \sqrt{\frac{1}{10\xi}(\varkappa\Omega^2-6\delta)\left[1-\sqrt{1-\frac{40\xi\left[4\beta-\left(\alpha-2\gamma A_b\sqrt{1-\frac{4\zeta^2}{\gamma^2A_b^2\Omega^2}}\right)\Omega^2\right]}{(6\delta-\varkappa\Omega^2)^2}}\right]} \quad (22)$$

and

$$A_2 = \sqrt{\frac{1}{10\xi}(\varkappa\Omega^2-6\delta)\left[1-\sqrt{1-\frac{40\xi\left[4\beta-\left(\alpha+2\gamma A_b\sqrt{1-\frac{4\zeta^2}{\gamma^2A_b^2\Omega^2}}\right)\Omega^2\right]}{(6\delta-\varkappa\Omega^2)^2}}\right]} \, . \quad (23)$$

The amplitude frequency relations, (22) and (23), are analyzed in the Appendix and compared to experimental values.

CHAPTER III

THE EXPERIMENTAL PROCEDURE

Experimental Apparatus

A thin rectangular column was mounted vertically on the table of a 25 lb. vibration fatigue testing machine which had independent displacement amplitude and frequency controls. The testing machine used in this experiment had an amplitude range of 0-0.075 inches and a frequency range of 0-100 cycles per second. Two rectangular blocks bolted to the table provided a vise grip to insure the column would act as a cantilever. A strain gage was attached near the column's base to measure strains and to determine damping constants for each mode of vibration.

The base amplitude was measured with a micrometer and base frequency was measured with a stroboscope. The signal from the strain gage was fed through a DC amplifier to a Tektronic 502A dual-beam oscilloscope.

Measurement of Strains

To compare experimental strains with calculated values the maximum strain was determined for various constant frequencies. The frequency was held constant at various values in the resonance region until a steady state response was observed. Immediately the maximum strain was read and recorded. The main problem encountered in taking data was the difficulty of holding a constant frequency of the vibration machine so a steady state solution with constant maximum amplitude would result. A slight drifting of the frequency had a great effect on the amplitude of vibration. However, with careful control, the experiments could be repreated and data could be reproduced satisfactorily.

Figure 2 illustrates the data taken in the laboratory. The stability bounds C and D were determined by increasing the frequency to a point C where the slightest disturbance of the column caused steady-state vibrations and by decreasing the frequency to a point D where small vibrations began. Strains were recorded for various values of the frequency in the resonance region to determine the curve ND. In addition a careful determination was made of the frequency value J, or cutoff point, at which any further decrease in frequency caused the response to jump to the identically zero solution.

CHAPTER IV

ANALYTICAL SOLUTION FOR STRAINS

Rather than measure amplitudes directly, the strains were recorded for a particular point on the column. Hence a relation between the analytical solution for the amplitude and the maximum strains was required.

To minimize the errors associated with computations for the strains using a solution which was only approximate, an external load system composed of the inertial forces was considered. The strains could then be calculated by static methods. This procedure eliminated the necessity of differentiating the displacements and gave a solution for the strains in terms of the displacements and their integrals. See [14].

The solutions for the displacements in the undamped case were

$$v(x,t) = A\psi_n(x) \sin \frac{\Omega}{2} t$$

and

$$u(x,t) = A_b \cos\Omega t - \frac{A^2 \sin^2 \frac{\Omega}{2} t}{2} \int_0^x (\psi_n')^2 \, dx \, .$$

Differentiating twice with respect to time yielded the accelerations

$$\ddot{v}(x,t) = - \frac{A\Omega^2}{4} \psi_n(x) \sin \frac{\Omega}{2} t$$

and

$$\ddot{u}(x,t) = - \left[A_b + \frac{A^2}{4} \int_0^x (\psi_n')^2 \, dx \right] \Omega^2 \cos\Omega t \, .$$

Hence the inertia loadings were

$$\frac{\rho A_c \Omega^2}{4} A\psi_n(x) \sin \frac{\Omega}{2} t$$

in the transverse direction and

$$\rho A_c \left[A_b + \frac{A^2}{4} \int_o^x (\psi_n')^2 \, dx\right] \Omega^2 \cos\Omega t$$

in the longitudinal direction. The moment at a point x = p, the location of the strain gage, due to the transverse loading was

$$M_t(p) = \frac{\rho A_c A \Omega^2}{4\beta_n^4} \psi_n''(p) \sin \frac{\Omega}{2} t .$$

The moment at x = p due to the longitudinal loading was

$$M_\ell(p) = \rho A_c \Omega^2 A\left[\int_p^\ell [\psi_n(x) - \psi_n(p)]\left[A_b + \frac{A^2}{4} \int_o^x (\psi_n')^2 dx\right] dx\right] \cos\Omega t .$$

An approximate relation

$$\sigma = Ee = \frac{My}{I}$$

or $e = \frac{My}{EI}$ was used to calculate strains. The coefficients of the trigonometric functions in the expressions for $M_t(p)$ and $M_\ell(p)$ were calculated and then separate plots were drawn for various values of the frequency and amplitude to determine the maximum moment. For the frequency range in which a response was found experimentally the moment due to longitudinal forces, $M_\ell(p)$, was found to be small in comparison to the moment, $M_t(p)$, due to transverse inertia forces. Hence the effect of $M_\ell(p)$ was neglected for values of the frequency and amplitude which were obtained experimentally.

For larger values of the amplitude which were found analytically at frequencies outside of the experimental region of response the effect of the moment due to longitudinal inertia forces became significant. However the plots drawn so $M_t(p)$ and $M_\ell(p)$ could be combined revealed an analytic solution which was not sinusoidal but experiments indicated the response should be almost sinusoidal. Hence no effort was made to describe strains analytically for values of the frequency and amplitude outside of the region found in experiments.

CHAPTER V

DISCUSSION OF RESULTS

Figures 5 and 6 based on experiments and the calculations of Chapter IV illustrate the results obtained in this investigation. The behavior of both experimental curves reflects the nature of the damping of the real system. For a constant steady-state amplitude damping decreases the region of frequencies for which motion occurs. Since linear damping had little effect on the amplitudes calculated in this investigation some other mechanism should be considered to obtain a more accurate approximation to the real problem. Bolotin [3] concludes this can be done by using a nonlinear damping term in equation (15).

Many approximations were made in the analysis but Figures 5 and 6 show a close agreement between theory and experiment. The deviation of the theory from the experiment becomes greater as the amplitude of vibration increases. This is expected since the effect of the moment due to longitudinal forces as calculated in Chapter IV becomes greater with increasing amplitude and this effect was neglected in the analysis. In addition a nonlinear damping term of the type described by Bolotin [3] has a greater effect as amplitude increases.

Significance of Fifth Order Term

Figures 7 and 8 show the calculated maximum transverse amplitudes of the vibrating column.

In the equation of motion, (13), the terms of fifth order were

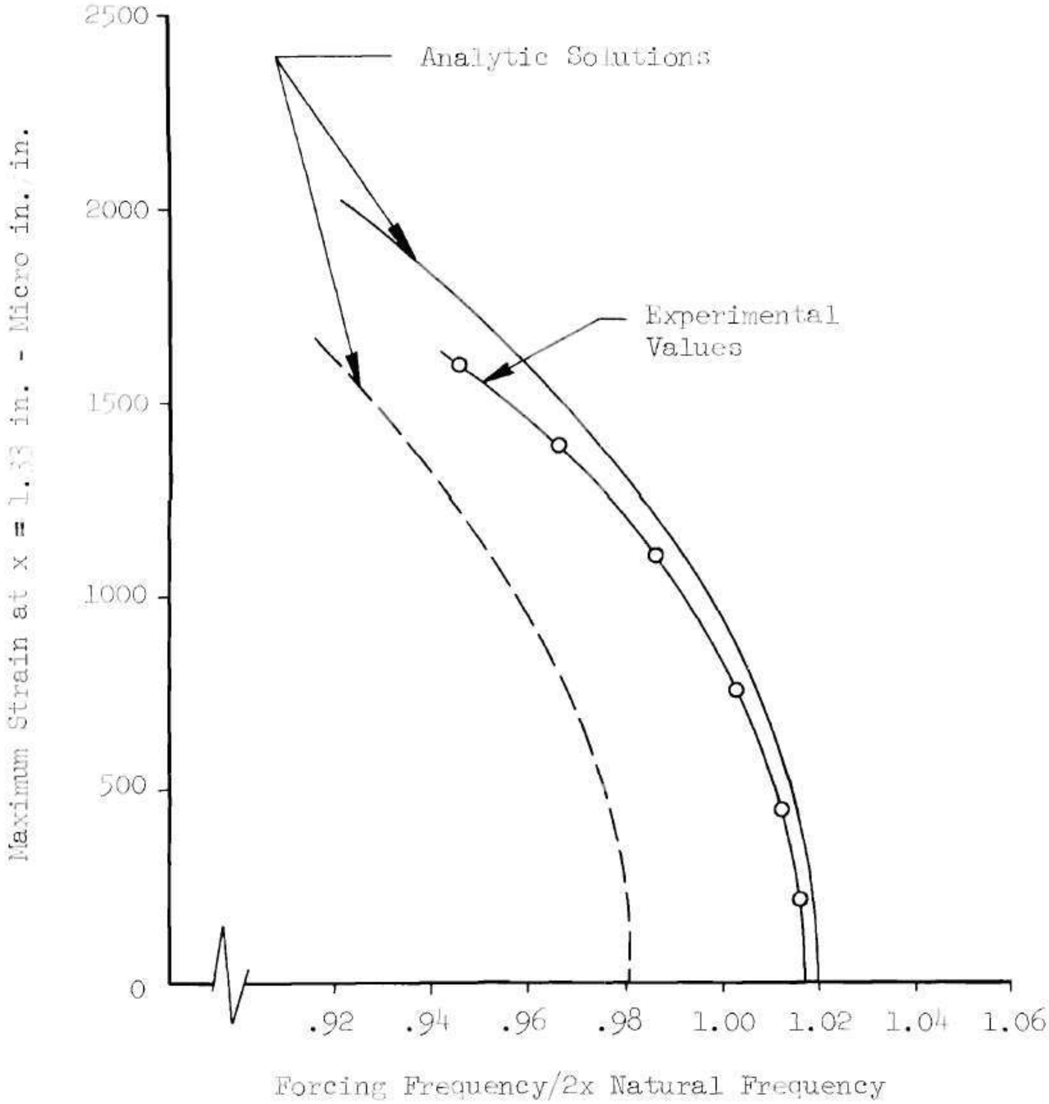

Figure 4. Maximum Strains for Second Mode.

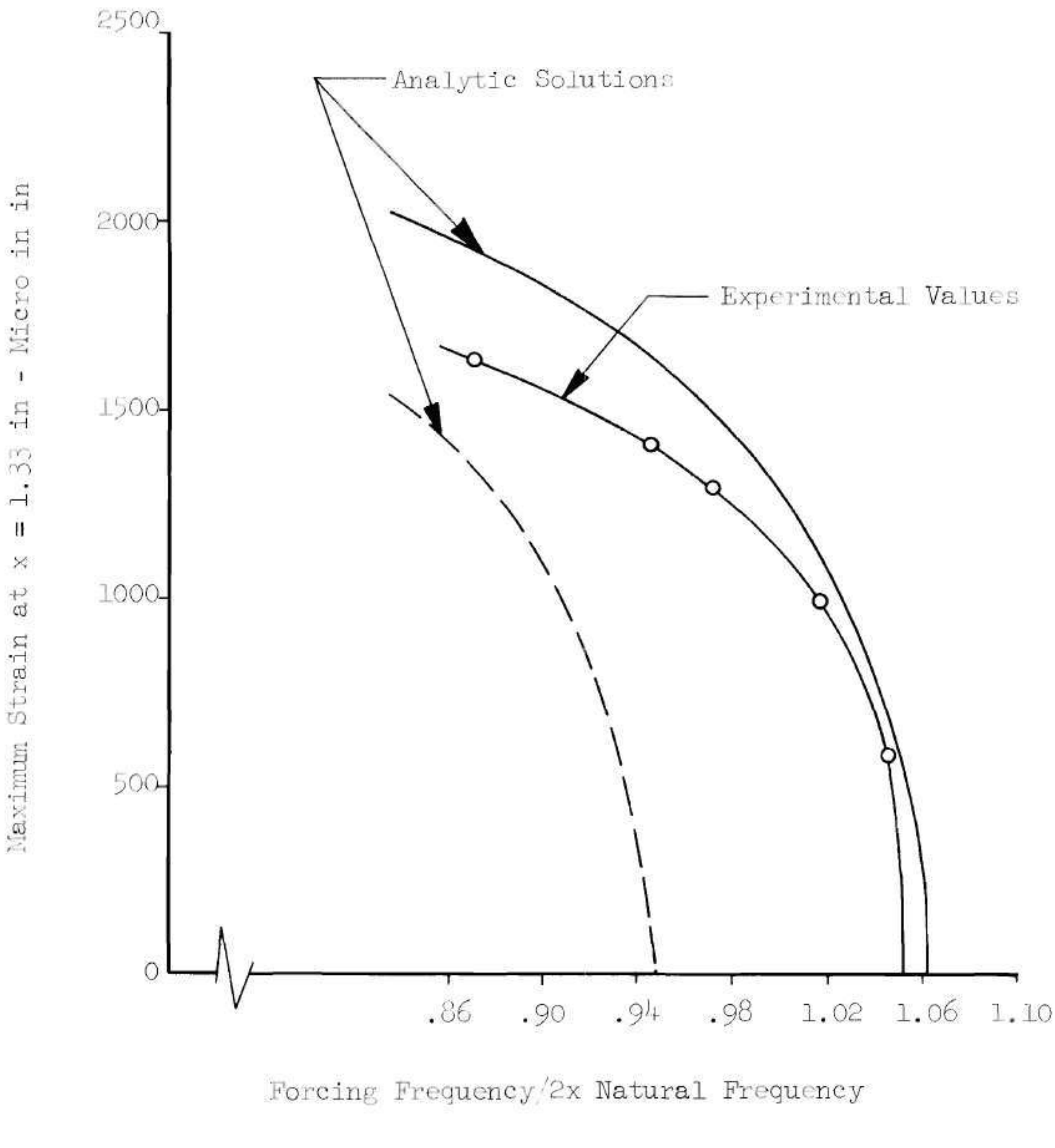

Figure 5. Maximum Strains for Third Mode.

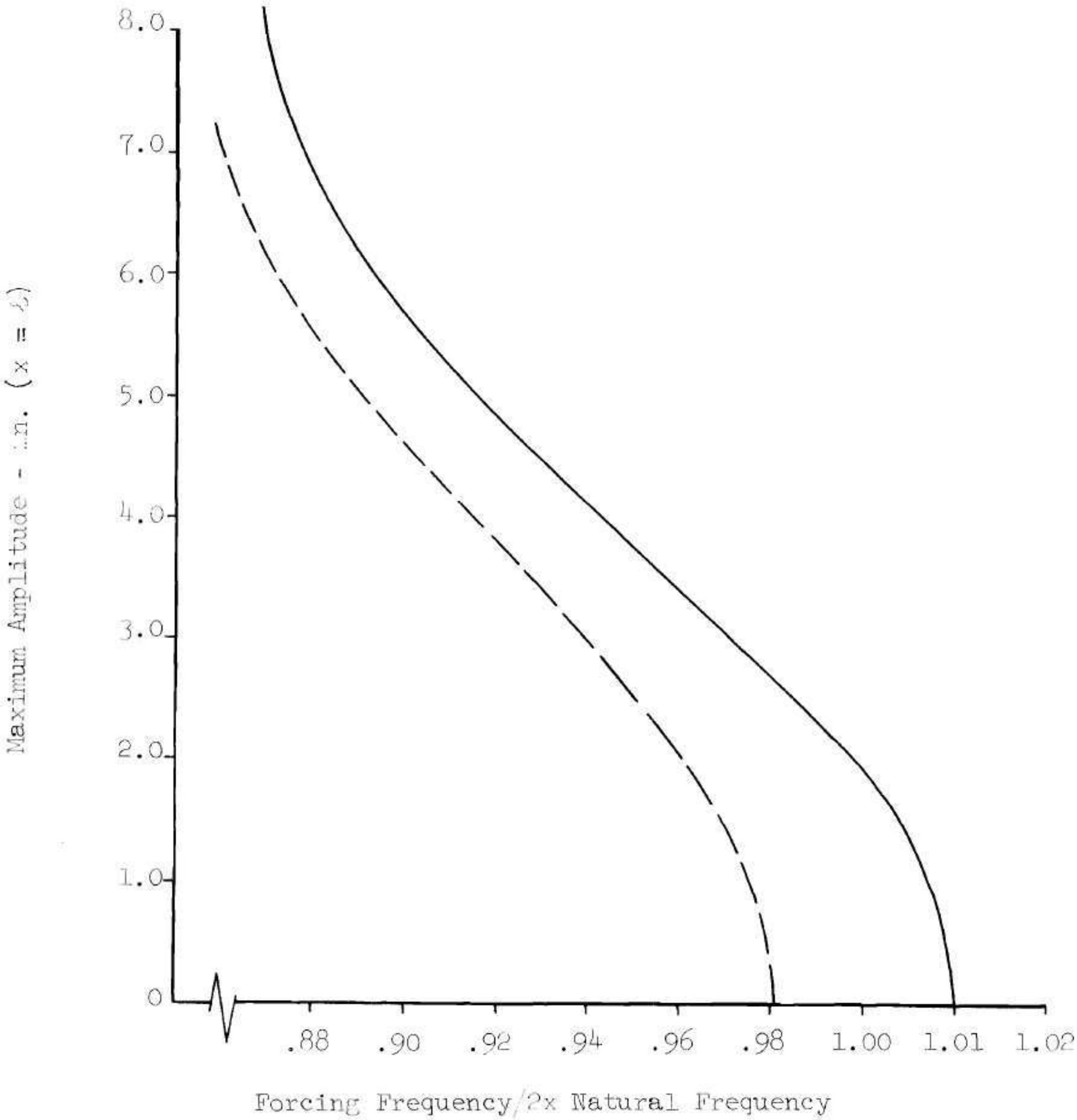

Figure 6. Amplitude Calculations for Second Mode.

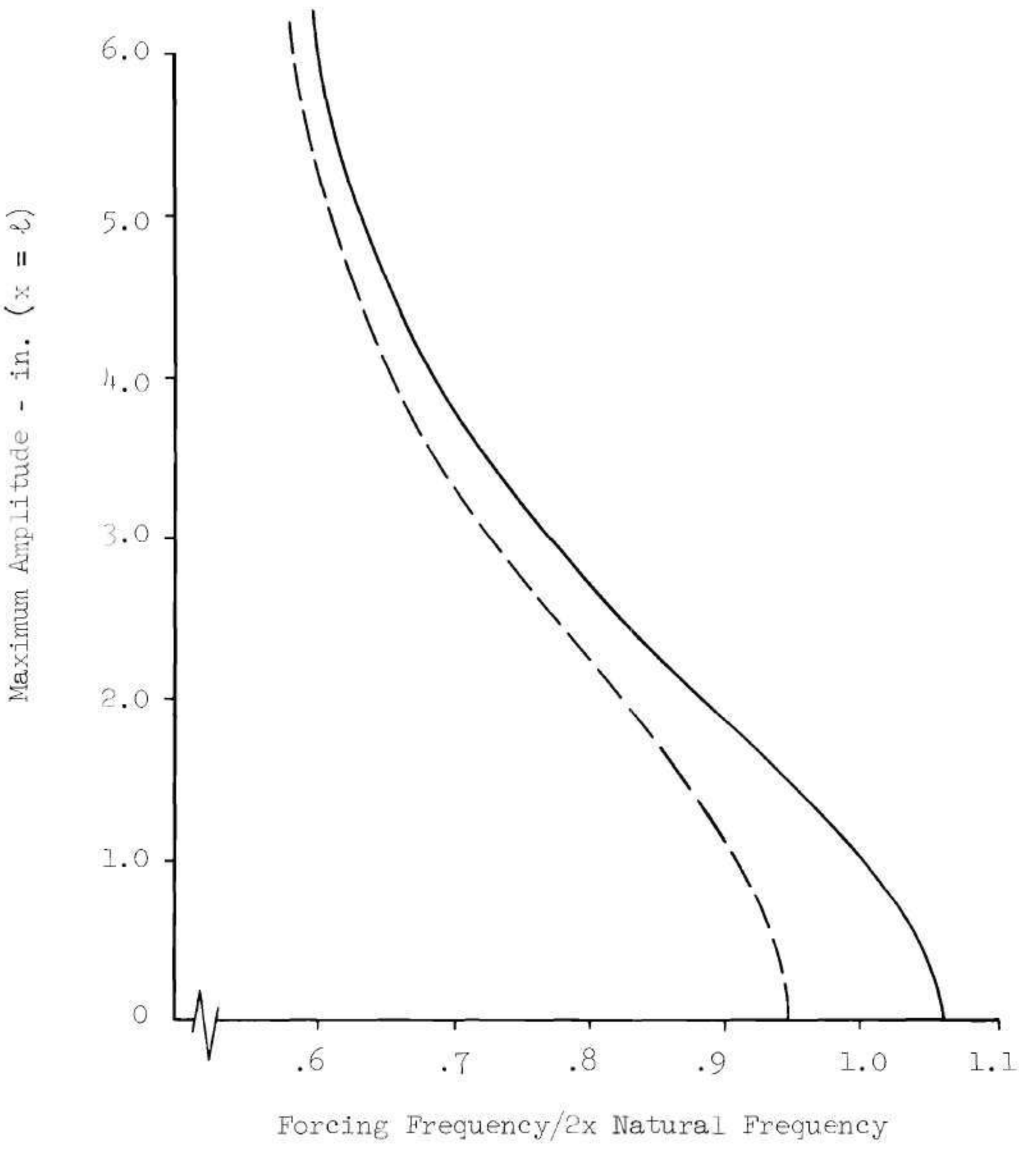

Figure 7. Amplitude Calculations for Third Mode.

kept because some of them compared in magnitude to terms of third order for slopes of the column of the order of 20°. The significance of the fifth order term also appears when analytically determining the cutoff point of vibrations. The values determined in this investigation differ significantly from the values determined by Bolotin's [3] equation

$$6\delta - \varkappa\Omega^2 = 0 \ .$$

The values determined here by equation (26) in the appendix yield higher and more realistic values. However Bolotin's response amplitudes differ little from the values calculated here for the experimental range of response frequencies. The fifth order term is important only for larger amplitude responses which are seldom found due to damping.

APPENDIX

EXPERIMENTAL OBSERVATIONS AND CALCULATIONS

Tables 1 and 2 give the data collected in the laboratory for the maximum strains.

Table 1. Experimental Strain
(Second Mode)

Length = 32.945 in.	Specific Weight = 485 lb/ft^3
Width = 1.0107 in.	Young's Modulus = 30×10^6 psi
Thickness = 0.0633 in.	Natural Frequency = 11.89 cps

$\frac{\text{Forcing Frequency}}{2 \times \text{Natural Frequency}}$	Strain Micro in/in
1.017	0
1.016	208
1.0126	458
1.009	562
1.0056	625
1.002	750
0.995	958
0.988	1125
0.981	1208
0.974	1292
0.9671	1375
0.9601	1458
0.9531	1562
0.9461	1604
0.940	0 - CUTOFF
0.9797	LOWER INSTABILITY BOUND

Table 2. Experimental Strain
(Third Mode)

Length = 32.945 in.	Specific Weight = 485 lb/ft^3
Width = 1.0107 in.	Young's Modulus = 30×10^6 psi
Thickness = 0.0633 in.	Natural Frequency = 33.48 cps

Forcing Frequency / 2 x Natural Frequency	Strain Micro in/in
1.051	0
1.045	583
1.02	1000
1.008	1083
0.971	1292
0.946	1417
0.921	1542
0.896	1583
0.871	1646
0.858	0 - CUTOFF
0.944	LOWER INSTABILITY BOUND

Amplitude Frequency Equation Analysis

The amplitude frequency equations from Chapter II are:

$$A_{1,2} = \sqrt{\frac{1}{10\xi}(\varkappa\Omega^2 - 6\delta)\left[1 - \sqrt{1 - \frac{40\xi\left[4\beta - \left(\alpha \pm 2\delta A_b\sqrt{1 - \frac{4\zeta^2}{\delta^2 A_b^2 \Omega^2}}\right)\Omega^2\right]}{(6\delta - \varkappa\Omega^2)^2}}\right]} . \quad \begin{matrix}(22)\\(23)\end{matrix}$$

These equations yielded the amplitudes for various frequencies and the linear stability bounds and cutoff frequencies. The linear stability bounds are the frequencies at which the above amplitudes are zero. That is when

$$4\beta - \left(\alpha \pm 2\gamma A_b\sqrt{1 - \frac{4\zeta^2}{\gamma^2 A_b^2 \Omega^2}}\right)\Omega^2 = 0$$

or

$$\Omega = 2\sqrt{\frac{(\alpha\beta - 2\zeta^2) \pm \sqrt{(2\zeta^2 - \alpha\beta)^2 - \beta^2(\alpha^2 - 4\gamma^2 A_b^2)}}{(\alpha^2 - 4\gamma^2 A_b^2)}} \ . \tag{24}$$

If damping and nonlinear terms are neglected in equation (16) of Chapter II and if the amplitude of the forcing frequency, A_b, is zero there results

$$\alpha\ddot{T} + \beta T = 0 \ . \tag{25}$$

The natural frequency of the system is then

$$\omega_n = \sqrt{\frac{\beta}{\alpha}} \ . \tag{26}$$

Neglecting damping in equation (24) and using (26), the stability bounds can be written in the form

$$\left(\frac{2\omega_n}{\Omega}\right)^2 = 1 \pm 2\,\frac{\gamma}{\alpha}\,A_b \ . \tag{27}$$

Equation (27) represents the first two terms in the series expansion for the stability bounds as determined by Burnside [2] from a Mathieu equation. Since the parameters used in the experiment were small, equation (27) was accurate enough to predict the stability bounds.

The cutoff frequency is reached when the frequency is reduced to a value such that

$$40\xi\left[4\beta - \left(\alpha \pm 2\gamma A_b\sqrt{1 - \frac{4\zeta^2}{\gamma^2 A_b^2 \Omega^2}}\right)\right] = (6\delta - \kappa\Omega^2)^2 . \qquad (28)$$

If the frequency is further reduced the inner radical in equations (22) and (23) becomes imaginary.

All of the above calculations were performed by a digital computer. The results of the computations are shown in Chapter V.

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